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Note

An Inequality for Derivatives of Polynomials with Positive Coefficients

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The present paper gives a complete answer to a question concerning an inequality for derivatives of polynomials with positive coefficients.

Let Π_N be the class of all algebraic polynomials of degree not greater than N, let $\mathscr{H}_N$ be the class of polynomials from Π_N which have only real coefficients and all of whose zeros lie in the half-plane $\mathrm{Re}(z) \leqslant 0$, and let Π_N^+ be the class of polynomials from Π_N with only nonnegative coefficients. Define

$$M(r, f) = \max_{|z| = r} |f(z)|.$$

It is well-known that the Bernstein inequality

$$M(r, f') \leqslant \frac{N}{r} M(r, f) \qquad (r > 0)$$

holds for $f(z) \in \Pi_N$.

Abi-khuzam [1] formulated a refined form of the Bernstein inequality for polynomials that involves the number of zeros in the disk relevant for the norm. First Abi-khuzam [1, p. 119] suggested the general form of the question that follows (in particular when $n(r, f) = N$ this is a conjecture of Erdős proved by Lax).

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Conjecture. Let $f \in \Pi_N$, $N > 0$, and let $n(r, f)$ be the counting function of its zeros in the disk $|z| < r$; that is, for $r > 0$, $n(r, f)$ equals the number of zeros of f in the disc $|z| < r$, where each zero is counted as many times as its multiplicity indicates. Then

$$M(r, f') \leqslant \frac{N + n(r, f)}{2r} M(r, f) \qquad (r > 0). \tag{1}$$

Then Abi-khuzam proved that the conjecture holds for $\mathscr{H}_N$.

THEOREM 1. *If* $f(z) \in \mathscr{H}_N$, $N > 0$, *then*

$$M(r, f') \leqslant \frac{N + n(r, f)}{2r} M(r, f) \qquad (r > 0).$$

Since Π_N^+ is a wider set than $\mathscr{H}_N$, in the last section of [1] Abi-khuzam asked: If the class $\mathscr{H}_N$ is replaced by Π_N^+, does the conclusion of Theorem 1 still hold for polynomials f in Π_N^+?

In answering this question, he constructed in [1] a polynomial $Q \in \Pi_5^+$,

$$Q(z) = z^5 + z^4 + 2z + 2,$$

and claimed that since $Q(z)$ does not satisfy inequality (1) this gives a negative answer to the question as well as to the above cited conjecture (see [1, p. 124]).

After careful verification, we found that there is something wrong in the counterexample given by Abi-khuzam. In fact, write

$$Q(z) = (z + 1)(z^4 + 2),$$

then evidently

$$\frac{Q'(z)}{Q(z)} = \frac{1}{z + 1} + \frac{4z^3}{z^4 + 2}.$$

A direct calculation leads to

$$\frac{4r^3}{r^4 + 2} \leqslant \begin{cases} \dfrac{2}{r}, & r \leqslant \sqrt[4]{2}, \\ \dfrac{4}{r}, & r > \sqrt[4]{2}, \end{cases}$$

then

$$\frac{M(r, Q')}{M(r, Q)} = \frac{Q'(r)}{Q(r)} \leqslant \begin{cases} \dfrac{3}{r}, & r \leqslant \sqrt[4]{2}, \\ \dfrac{5}{r}, & r > \sqrt[4]{2}. \end{cases}$$

That is, $Q(z)$ still satisfies (1). This allows us to repeat the question raised by Abi-khuzam again: Is there a similar estimate to (1) for $f \in \Pi_N^+$?

The present note will establish the following result to give this question a complete solution, which also gives a correct negative answer to the above cited conjecture.

THEOREM 2. (i) *If* $f \in \Pi_N^+$ *for* $N = 1, 2, 3, 4$, *then*

$$M(r, f') \leqslant \frac{N + n(r, f)}{2r} M(r, f) \qquad (r > 0).$$

(ii) *Let* $5m \leqslant N < 5(m+1)$, $m = 1, 2, \ldots$ *Then there exists a polynomial* $f_N \in \Pi_N^+$ *such that*

$$M(3, f') = \frac{N + n(3, f) + 2m/7}{2 \times 3} M(3, f) > \frac{N + n(3, f)}{2 \times 3} M(3, f).$$

Proof. (i) Since the argument of this part is completely elementary, we omit it here.

(ii) Define

$$f_5(z) = (z^2 - z + 1)(z + 3)^3 = z^5 + 8z^4 + 19z^3 + 9z^2 + 27,$$

then

$$f_5'(z) = 5z^4 + 32z^3 + 57z^2 + 18z.$$

We check that

$$\frac{M(3, f_5')}{M(3, f_5)} = \frac{f_5'(3)}{f_5(3)} = \frac{1836}{1512} = \frac{7 + 2/7}{2 \times 3}. \tag{2}$$

If $N = 5m$, $m = 2, 3, 4, \ldots$, let

$$f_N(z) = f_5^m(z).$$

Hence

$$\frac{M(3, f_N')}{M(3, f_N)} = \frac{m f_5'(3)}{f_5(3)} = \frac{m(7 + 2/7)}{2 \times 3} = \frac{N + n(3, f_N) + 2m/7}{2 \times 3}.$$

Finally, for $5m < N < 5(m+1)$, $m = 1, 2, \ldots$, set

$$f_N(z) = f_{5m}(z)(z+3)^{N-5m}.$$

From (2), we calculate that

$$\frac{M(3, f'_N)}{M(3, f_N)} = \frac{f'_{5m}(3)}{f_{5m}(3)} + \frac{N-5m}{6} = \frac{N+2m+2m/7}{2\times 3} = \frac{N+n(3, f_N)+2m/7}{2\times 3}.$$

Thus we are done. ▮

Reference

1. F. F. Abi-Khuzam, An inequality for derivatives of polynomials whose zeros are in a half-plane, *Proc. Amer. Math. Soc.* **89** (1983), 119–124.